LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

Volume and surface area of a spherical harmonic surface approximation to a NIF implosion core defined by HGXI/GXD images from the equator and pole

J. A. Koch

November 2, 2011

Disclaimer

This document was prepared as an account of work sponsored by an agency of the United States government. Neither the United States government nor Lawrence Livermore National Security, LLC, nor any of their employees makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States government or Lawrence Livermore National Security, LLC. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States government or Lawrence Livermore National Security, LLC, and shall not be used for advertising or product endorsement purposes.

This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

Volume and surface area of a spherical harmonic surface approximation to a NIF implosion core defined by HGXI/GXD images from the equator and pole

A solid object, such as a simplified approximation to an implosion core defined by the 17% intensity contour, can be described by a sum of spherical harmonics, following the notation of Butkov (Mathematical Physics, ISBN 0-201-00727-4, 1968; there are other notations so care is required):

$$r(\theta,\phi) = \sum_{l=0}^{\infty} \left[A_{l0} Y_{l0}(\theta,\phi) + \sum_{m=1}^{l} \left[A_{lm}^+ Y_{lm}^+(\theta,\phi) + A_{lm}^- Y_{lm}^-(\theta,\phi) \right] \right]$$

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}} P_l(\cos\theta)$$

$$Y_{lm}^+(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{2\pi(l+m)!}} P_l^m(\cos\theta)\cos m\phi$$

$$Y_{lm}^-(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{2\pi(l+m)!}} P_l^m(\cos\theta)\sin m\phi$$

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m P_l(x)}{dx^m}$$

with $P_l(x)$ being the usual (apparently standard) Legendre polynomial. For the present purposes, finding the volume and surface area of an implosion core defined by P0, P2, P4, M0, and M4, I will restrict the problem to consider only A_{00}, A_{20}, A_{40}, and A_{44}, with the phase angle set to eliminate the $\sin(m\phi)$ term. Once the volume and surface area are determined, I will explore how these coefficients relate to measured quantities A0, A2/A0, A4/A0, M0, and M4/M0.

Using the above definitions, I can explicitly write the functions of interest:

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_4^4(x) = 105(1 - x^2)^2$$

$$Y_{00}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_{20}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$$

$$Y_{40}(\theta,\phi) = \frac{3}{16}\sqrt{\frac{1}{\pi}}(35\cos^4\theta - 30\cos^2\theta + 3)$$

$$Y_{44}(\theta,\phi) = \frac{3}{16}\sqrt{\frac{35}{\pi}}\sin^4\theta\cos 4\phi$$

Therefore, I can write $r(\theta,\phi)$ explicitly as well:

$$r(\theta,\phi) = \frac{1}{2\sqrt{\pi}}\left[A_{00} - \frac{\sqrt{5}}{2}A_{20} + \frac{9}{8}A_{40}\right] + \frac{3}{4}\sqrt{\frac{5}{\pi}}\left[A_{20} - \frac{15}{2\sqrt{5}}A_{40}\right]\cos^2\theta + \frac{105A_{40}}{16\sqrt{\pi}}\cos^4\theta + \frac{3A_{44}}{16}\sqrt{\frac{35}{\pi}}\sin^4\theta\cos 4\phi$$

$$\equiv A + B\cos^2\theta + C\cos^4\theta + D\sin^4\theta\cos 4\phi$$

The volume enclosed within a surface defined by $r(\theta,\phi)$, and the area of that surface (http://ciks.cbt.nist.gov/~garbocz/paper134/node4.html), can be written as:

$$V = \frac{1}{3}\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi(r(\theta,\phi))^3$$

$$A = \int_0^\pi d\theta\int_0^{2\pi}d\phi\, r(\theta,\phi)\sqrt{\left(\frac{\partial r(\theta,\phi)}{\partial\phi}\right)^2 + \left(\frac{\partial r(\theta,\phi)}{\partial\theta}\right)^2\sin^2\theta + (r(\theta,\phi))^2\sin^2\theta}$$

I. Volume

The volume integral can be written as:

$$V = \frac{2\pi}{3}\int_0^\pi d\theta\sin\theta\left[\begin{array}{l}A^3 + 3A^2B\cos^2\theta + 3A(AC + B^2)\cos^4\theta + B(6AC + B^2)\cos^6\theta \\ +3C(AC + B^2)\cos^8\theta + 3BC^2\cos^{10}\theta + C^3\cos^{12}\theta\end{array}\right]$$

$+$

$$\frac{D}{3}\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin^5\theta\cos 4\phi[\text{terms involving only } \theta]$$

$+$

$$\frac{D^2}{3}\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin^9\theta\cos^2 4\phi\left[3A + 3B\cos^2\theta + 3C\cos^4\theta\right]$$

$+$

$$\frac{D^3}{3}\int_0^\pi d\theta\int_0^{2\pi}d\phi\sin^{13}\theta\cos^3 4\phi$$

The second and fourth terms vanish, and the remainder can be written as:

$$V = \frac{4\pi}{3}\left[\begin{array}{l}A^3 + A^2B + 3A(AC + B^2)/5 + B(6AC + B^2)/7 \\ +3C(AC + B^2)/9 + 3BC^2/11 + C^3/13\end{array}\right]$$

$+$

$$\frac{\pi D^2}{3}\int_0^\pi d\theta\sin^9\theta\left[3A + 3B\cos^2\theta + 3C\cos^4\theta\right]$$

The second term can be written as:

$$\frac{\pi D^2}{3}\left[3(A+B+C)\int_0^{\pi} \sin^9\theta d\theta - 3(B+2C)\int_0^{\pi}\sin^{11}\theta d\theta + 3C\int_0^{\pi}\sin^{13}\theta d\theta\right]$$

$$=\frac{2\pi D^2}{3}\left[3(A+B+C)\frac{8*6*4*2}{9*7*5*3} - 3(B+2C)\frac{10*8*6*4*2}{11*9*7*5*3} + 3C\frac{12*10*8*6*4*2}{13*11*9*7*5*3}\right]$$

$$=\frac{256\pi D^2}{315}\left[A+\frac{B}{11}+\frac{3C}{143}\right]$$

so the complete volume integral becomes:

$$V = \frac{4\pi}{3}\left[\begin{array}{l} A^3 + A^2 B + \left(\frac{3A}{5}+\frac{C}{3}\right)(AC+B^2) + \frac{B(6AC+B^2)}{7} \\[2mm] +C^2\left(\frac{3B}{11}+\frac{C}{13}\right) + \frac{64D^2}{105}\left(A+\frac{B}{11}+\frac{3C}{143}\right) \end{array}\right]$$

Now, with experimental data we fit 2D implosion images to Legendre polynomials and Fourier modes from the equatorial and polar views respectively:

$$r(\theta, \text{averaged over } \phi) = A_0 + \frac{A_2}{2}(3\cos^2\theta - 1) + \frac{A_4}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$$

$$= A_0(1-\frac{P_2}{2}+\frac{3P_4}{8}) + \frac{3A_0}{2}(P_2 - \frac{5P_4}{2})\cos^2\theta + \frac{35A_0}{8}P_4\cos^2\theta$$

$$r(\theta = 90^0, \phi) = m_0 + m_4\cos 4\phi$$

Considering the full spherical harmonic shape defined above in terms of A, B, C and D, and neglecting the impact of m4 on the equatorial view (which averages to zero, but generally will depend on the azimuth of the equatorial view), we can identify the terms as:

$$A = A_0(1 - \frac{P_2}{2} + \frac{3P_4}{8})$$

$$B = \frac{3A_0}{2}(P_2 - \frac{5P_4}{2})$$

$$C = \frac{35A_0}{8}P_4$$

$$D = m_0 M_4 = A_0(1 - \frac{P_2}{2} + \frac{3P_4}{8})M_4$$

Finally, separating out A0 and rewriting, we have:

$$\alpha = (1 - \frac{P_2}{2} + \frac{3P_4}{8})$$

$$\beta = \frac{3}{2}(P_2 - \frac{5P_4}{2})$$

$$\gamma = \frac{35}{8}P_4$$

$$\delta = (1 - \frac{P_2}{2} + \frac{3P_4}{8})M_4$$

$$V = \frac{4\pi A_0^3}{3}\left[\begin{array}{l} \alpha^2(\alpha + \beta) + \left(\frac{3\alpha}{5} + \frac{\gamma}{3}\right)(\alpha\gamma + \beta^2) + \frac{\beta(6\alpha\gamma + \beta^2)}{7} \\ + \gamma^2\left(\frac{3\beta}{11} + \frac{\gamma}{13}\right) + \frac{64\delta^2}{105}\left(\alpha + \frac{\beta}{11} + \frac{3\gamma}{143}\right) \end{array}\right]$$

This is a cumbersome equation to manipulate, but it is simple to program into a calculator or an Excel spreadsheet.

I. Surface Area

The surface area integral will in general be impossible to solve analytically, so from the start I will aim to solve it numerically. The integral is:

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi\, r(\theta,\phi)\sqrt{\left(\frac{\partial r(\theta,\phi)}{\partial\phi}\right)^2 + \left(\frac{\partial r(\theta,\phi)}{\partial\theta}\right)^2 \sin^2\theta + (r(\theta,\phi))^2 \sin^2\theta}$$

Taking the derivatives and rearranging, this is:

$$A = A_0^2\left[\begin{array}{l} \int_0^\pi d\theta \int_0^{2\pi} d\phi\left(\alpha + \beta\cos^2\theta + \gamma\cos^4\theta + \delta\sin^4\theta\cos 4\phi\right) \\ \sqrt{\begin{array}{l} 16\delta^2\sin^8\theta\sin^2 4\phi + 4\sin^4\theta\cos^2\theta(\beta + 2\gamma\cos^2\theta - 2\delta\sin^2\theta\cos 4\phi)^2 + \\ \sin^2\theta\left(\alpha + \beta\cos^2\theta + \gamma\cos^4\theta + \delta\sin^4\theta\cos 4\phi\right)^2 \end{array}} \end{array}\right]$$

4

where α, β, γ and δ are defined above. This equation can be solved numerically in various ways, and a FORTRAN code example is shown below:

```fortran
      program surfacevolume
      pi = acos(-1.0)
c
      a0 = 20.879
      p2 = -0.212
      p4 = 0.067
      xm4 = 0.112
      alpha = 1.0-p2/2.0+3.0*p4/8.0
      beta = 3.0*(p2-5.0*p4/2.0)/2.0
      gamma = 35.0*p4/8.0
      delta = (1.0-p2/2.0+3.0*p4/8.0)*xm4
c
      n = 1000
      m = 1000
      deltax = pi/n
      deltay = 2.0*pi/m
      area = 0.0
c
      do 100 i=1,n-1
        x = i*deltax
        do 200 j=1,m-1
          y = j*deltay
          r = alpha+beta*cos(x)**2+gamma*cos(x)**4+delta*sin(x)**4*
     >     cos(4.0*y)
          dphi = -4.0*delta*sin(x)**4*sin(4.0*y)
          dth = -2.0*sin(x)*cos(x)*(beta+2.0*gamma*cos(x)**2-
     >     2.0*delta*sin(x)**2*cos(4.0*y))
          area = area + r*sqrt(dphi**2+dth**2*sin(x)**2+r**2*
     >     sin(x)**2)
200     continue
100   continue
      area = area*deltax*deltay/(4.0*pi)
c
      volume = alpha**2*(alpha+beta)+(3.0*alpha/5.0+gamma/3.0)*
     > (alpha*gamma+beta**2)+beta*(6.0*alpha*gamma+beta**2)/7.0+
     > gamma**2*(3.0*beta/11.0+gamma/13.0)+64.0*delta**2*(alpha+
     > beta/11.0+3.0*gamma/143.0)/105.0
      write(6,*) 'volume = ',volume*(4.0*pi*a0**3/3.0),
     > 'volume multiplier = ',volume,' radius multipler = ',
     > volume**0.3333333
      write(6,*) 'surface area = ',area*4.0*pi*a0**2,
     > 'area multiplier = ',area,' radius multipler = ',area**0.5
c
      pause 'hit cr'
      end
```

The numerical results from this program agree with analytical and numerical results I have previously obtained considering only P2, and also considering only M4.

III. Relationship to spherical harmonics coefficients

Finally, I can write the spherical harmonics coefficients defined above in terms of the measured quantities:

$$A_{00} = 2\sqrt{\pi}A_0 \approx 3.545A_0$$

$$A_{20} = \frac{2\sqrt{\pi}}{\sqrt{5}} A_2 \approx 1.585A_0P_2$$

$$A_{40} = \frac{2\sqrt{\pi}}{3} A_4 \approx 1.182A_0P_4$$

$$A_{44} = \frac{16\sqrt{\pi}}{3\sqrt{35}}(A_0 - \frac{A_2}{2} + \frac{3A_4}{8})M4 \approx 1.598A_0(1 - 0.5P_2 + 0.375P_4)M4$$